GLOBAL EDITION

Linear Algebra with Applications

NINTH EDITION

Steven J. Leon

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Linear Algebra with Applications

Ninth Edition

Global Edition

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To the memories of

Florence and Rudolph Leon,
devoted and loving parents

and to the memories of

Gene Golub, Germund Dahlquist, and Jim Wilkinson,
friends, mentors, and role models

Contents

* Online: The supplemental Chapters 8 and 9 can be downloaded from the Internet. See the section of the Preface on supplementary materials.

Preface

I am pleased to see the text reach its ninth edition. The continued support and enthusiasm of the many users has been most gratifying. Linear algebra is more exciting now than at almost any time in the past. Its applications continue to spread to more and more fields. Largely due to the computer revolution of the last 75 years, linear algebra has risen to a role of prominence in the mathematical curriculum rivaling that of calculus. Modern software has also made it possible to dramatically improve the way the course is taught.

The first edition of this book was published in 1980. Many significant changes were made for the second edition (1986), most notably the exercise sets were greatly expanded and the linear transformations chapter of the book was completely revised. Each of the following editions has seen significant modifications including the addition of comprehensive sets of MATLAB computer exercises, a dramatic increase in the number of applications, and many revisions in the various sections of the book. I have been fortunate to have had outstanding reviewers and their suggestions have led to many important improvements in the book. For the ninth edition we have given special attention to Chapter 7 as it is the only chapter that has not seen major revisions in any of the previous editions. The following is an outline of the most significant revisions that were made for the ninth edition.

What's New in the Ninth Edition?

1. New Subsection Added to Chapter 3
Section 2 of Chapter 3 deals with the topic of subspaces. One important example of a subspace occurs when we find all solutions to a homogeneous system of linear equations. This type of subspace is referred to as a *null space*. A new subsection has been added to show how the null space is also useful in finding the solution set to a nonhomogeneous linear system. The subsection contains a new theorem and a new figure that provides a geometric illustration of the theorem. Three related problems have been added to the exercises at the end of Section 2.

2. New Applications Added to Chapters 1, 5, 6, and 7
In Chapter 1, we introduce an important application to the field of Management Science. Management decisions often involve making choices between a number of alternatives. We assume that the choices are to be made with a fixed goal in mind and should be based on a set of evaluation criteria. These decisions often involve a number of human judgments that may not always be completely consistent. The analytic hierarchy process is a technique for rating the various alternatives based on a chart consisting of weighted criteria and ratings that measure how well each alternative satisfies each of the criteria.

In Chapter 1, we see how to set up such a chart or decision tree for the process. After weights and ratings have been assigned to each entry in the chart, an overall ranking of the alternatives is calculated using simple matrix-vector operations. In Chapters 5 and 6, we revisit the application and discuss how to use advanced matrix techniques to determine appropriate weights and ratings for the decision process. Finally in Chapter 7, we present a numerical algorithm for computing the weight vectors used in the decision process.

3. Section 1 of Chapter 7 Revised and Two Subsections Added
   Section 7.1 has been revised and modernized. A new subsection on IEEE floating-point representation of numbers and a second subsection on accuracy and stability of numerical algorithms have been added. New examples and additional exercises on these topics are also included.
4. Section 5 of Chapter 7 Revised
   The discussion of Householder transformations has been revised and expanded. A new subsection has been added, which discusses the practicalities of using QR factorizations for solving linear systems. New exercises have also been added to this section.
5. Section 7 of Chapter 7 Revised
   Section 7.7 deals with numerical methods for solving least squares problems. The section has been revised and a new subsection on using the modified Gram–Schmidt process to solve least squares problems has been added. The subsection contains one new algorithm.

Overview of Text

This book is suitable for either a sophomore-level course or for a junior/senior level course. The student should have some familiarity with the basics of differential and integral calculus. This prerequisite can be met by either one semester or two quarters of elementary calculus.

If the text is used for a sophomore-level course, the instructor should probably spend more time on the early chapters and omit many of the sections in the later chapters. For more advanced courses, a quick review of the topics in the first two chapters and then a more complete coverage of the later chapters would be appropriate. The explanations in the text are given in sufficient detail so that beginning students should have little trouble reading and understanding the material. To further aid the student, a large number of examples have been worked out completely. Additionally, computer exercises at the end of each chapter give students the opportunity to perform numerical experiments and try to generalize the results. Applications are presented throughout the book. These applications can be used to motivate new material or to illustrate the relevance of material that has already been covered.

The text contains all the topics recommended by the National Science Foundation (NSF) sponsored Linear Algebra Curriculum Study Group (LACSG) and much more. Although there is more material than can be covered in a one-quarter or one-semester course, it is my feeling that it is easier for an instructor to leave out or skip material than it is to supplement a book with outside material. Even if many topics are omitted, the book should still provide students with a feeling for the overall scope of the subject matter. Furthermore, students may use the book later as a reference and consequently may end up learning omitted topics on their own.

In the next section of this preface, a number of outlines are provided for one-semester courses at either the sophomore level or the junior/senior level and with either a matrix-oriented emphasis or a slightly more theoretical emphasis.

Ideally, the entire book could be covered in a two-quarter or two-semester sequence. Although two semesters of linear algebra has been recommended by the LACSG, it is still not practical at many universities and colleges. At present there is no universal agreement on a core syllabus for a second course. Indeed, if all of the topics that instructors would like to see in a second course were included in a single volume, it would be a weighty book. An effort has been made in this text to cover all of the basic linear algebra topics that are necessary for modern applications. Furthermore, two additional chapters for a second course are available for downloading from the special Pearson Web site developed for this book:

http://pearsonglobaleditions.com/leon

Suggested Course Outlines

I. Two-Semester Sequence: In a two-semester sequence, it is possible to cover all 40 sections of the book. When the author teaches the course, he also includes an extra lecture demonstrating how to use the MATLAB software.

II. One-Semester Sophomore-Level Course

A. A Basic Sophomore-Level Course

Chapter 1	Sections 1–6		7 lectures
Chapter 2	Sections 1–2		2 lectures
Chapter 3	Sections 1–6		9 lectures
Chapter 4	Sections 1–3		4 lectures
Chapter 5	Sections 1–6		9 lectures
Chapter 6	Sections 1–3		4 lectures
		Total	35 lectures

B. The LACSG Matrix Oriented Course: The core course recommended by the Linear Algebra Curriculum Study Group involves only the Euclidean vector spaces. Consequently, for this course you should omit Section 1 of Chapter 3 (on general vector spaces) and all references and exercises involving function spaces in Chapters 3 to 6. All of the topics in the LACSG core syllabus are included in the text. It is not necessary to introduce any supplementary materials. The LACSG recommended 28 lectures to cover the core material. This is possible if the class is taught in lecture format with an additional recitation section meeting once a week. If the course is taught without recitations, it is my feeling that the following schedule of 35 lectures is perhaps more reasonable.

Chapter 1	Sections 1–6		7 lectures
Chapter 2	Sections 1–2		2 lectures
Chapter 3	Sections 2–6		7 lectures
Chapter 4	Sections 1–3		2 lectures
Chapter 5	Sections 1–6		9 lectures
Chapter 6	Sections 1,3–5		8 lectures
		Total	35 lectures

III. One-Semester Junior/Senior Level Courses: The coverage in an upper division course is dependent on the background of the students. Below are two possible courses with 35 lectures each.

A. Course 1

Chapter 1	Sections 1–6	6 lectures
Chapter 2	Sections 1–2	2 lectures
Chapter 3	Sections 1–6	7 lectures
Chapter 5	Sections 1–6	9 lectures
Chapter 6	Sections 1–7	10 lectures
	Section 8 if time allows	
Chapter 7	Section 4	1 lecture

B. Course 2

Review of Topics in Chapters 1–3		5 lectures
Chapter 4	Sections 1–3	2 lectures
Chapter 5	Sections 1–6	10 lectures
Chapter 6	Sections 1–7	11 lectures
	Section 8 if time allows	
Chapter 7	Sections 4–7	7 lectures
	If time allows, Sections 1–3	

Computer Exercises

This edition contains a section of computing exercises at the end of each chapter. These exercises are based on the software package MATLAB. The MATLAB Appendix in the book explains the basics of using the software. MATLAB has the advantage that it is a powerful tool for matrix computations and yet it is easy to learn. After reading the Appendix, students should be able to do the computing exercises without having to refer to any other software books or manuals. To help students get started, we recommend one 50-minute classroom demonstration of the software. The assignments can be done either as ordinary homework assignments or as part of a formally scheduled computer laboratory course.

Another source of MATLAB exercises for linear algebra is the ATLAST book, which is available as a companion manual to supplement this book. (See the list of supplementary materials in the next section of this preface.)

While the course can be taught without any reference to the computer, we believe that computer exercises can greatly enhance student learning and provide a new dimension to linear algebra education. One of the recommendations of the Linear Algebra Curriculum Study Group is that technology should be used in a first course in linear algebra. That recommendation has been widely accepted, and it is now common to see mathematical software packages used in linear algebra courses.

Supplementary Materials

Web Supplements and Additional Chapters

Two supplemental chapters for this book may be downloaded using links from the Pearson Web site for this book:

http://pearsonglobaleditions.com/leon

The additional chapters are:

- Chapter 8. Iterative Methods
- Chapter 9. Canonical Forms

The Pearson Web site for this book contains materials for students and instructors including links to online exercises for each of the original seven chapters of the book, and a link to the errata list for this textbook. Please send any additional errata items that you discover to the author so that the list can be updated and corrections can be made in later printings of the book.

Companion Books

A number of MATLAB and Maple computer manuals are available as companion books. To obtain information about the companion packages available, instructors should either consult their Pearson sales representative or search the instructor section of the Pearson higher education Web site *www.pearsonglobaleditions.com/leon*. The following is a list of some of the companion books being offered as bundles with this textbook:

- *ATLAST Computer Exercises for Linear Algebra, Second Edition*. ATLAST (Augmenting the Teaching of Linear Algebra through the use of Software Tools) was an NSF-sponsored project to encourage and facilitate the use of software in the teaching of linear algebra. During a five-year period, 1992–1997, the ATLAST Project conducted 18 faculty workshops using the MATLAB software package. Participants in those workshops designed computer exercises, projects, and lesson plans for software-based teaching of linear algebra. A selection of these materials was first published as a manual in 1997. That manual was greatly expanded for the second edition published in 2003. Each of the eight chapters in the second edition contains a section of short exercises and a section of longer projects.

The collection of software tools (M-files) developed to accompany the ATLAST book may be downloaded from the ATLAST Web site:

www1.umassd.edu/specialprograms/atlast

Additionally, Mathematica users can download the collection of *ATLAST Mathematica Notebooks* that has been developed by Richard Neidinger.

- *Linear Algebra Labs with MATLAB: 3rd ed.* by David Hill and David Zitarelli
- *Visualizing Linear Algebra using Maple*, by Sandra Keith
- *A Maple Supplement for Linear Algebra*, by John Maloney
- *Understanding Linear Algebra Using MATLAB*, by Erwin and Margaret Kleinfeld

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I would like to acknowledge the contributions of Gene Golub and Jim Wilkinson. Most of the first edition of the book was written in 1977–1978 while I was a Visiting Scholar at Stanford University. During that period, I attended courses and lectures on numerical linear algebra given by Gene Golub and J. H. Wilkinson. Those lectures have greatly influenced me in writing this book. Finally, I would like to express my gratitude to Germund Dahlquist for his helpful suggestions on earlier editions of the book. Although Gene Golub, Jim Wilkinson, and Germund Dahlquist are no longer with us, they continue to live on in the memories of their friends.

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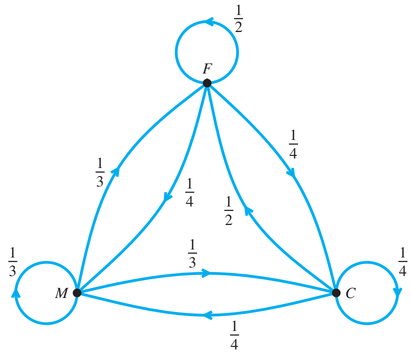
CHAPTER

1



Matrices and Systems of Equations

Probably the most important problem in mathematics is that of solving a system of linear equations. Well over 75 percent of all mathematical problems encountered in scientific or industrial applications involve solving a linear system at some stage. By using the methods of modern mathematics, it is often possible to take a sophisticated problem and reduce it to a single system of linear equations. Linear systems arise in applications to such areas as business, economics, sociology, ecology, demography, genetics, electronics, engineering, and physics. Therefore, it seems appropriate to begin this book with a section on linear systems.

1.1 Systems of Linear Equations

A *linear equation in n unknowns* is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \ldots, a_n$ and b are real numbers and $x_1, x_2, \ldots, x_n$ are variables. A *linear system* of m equations in n unknowns is then a system of the form

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned} \tag{1}$$

where the a_{ij}'s and the b_i's are all real numbers. We will refer to systems of the form (1) as $m \times n$ linear systems. The following are examples of linear systems:

(a) $\begin{aligned} x_1 + 2x_2 &= 5 \\ 2x_1 + 3x_2 &= 8 \end{aligned}$ **(b)** $\begin{aligned} x_1 - x_2 + x_3 &= 2 \\ 2x_1 + x_2 - x_3 &= 4 \end{aligned}$ **(c)** $\begin{aligned} x_1 + x_2 &= 2 \\ x_1 - x_2 &= 1 \\ x_1 \phantom{{}+x_2} &= 4 \end{aligned}$

System **(a)** is a 2×2 system, **(b)** is a 2×3 system, and **(c)** is a 3×2 system.

By a solution of an $m \times n$ system, we mean an ordered n-tuple of numbers $(x_1, x_2, \ldots, x_n)$ that satisfies all the equations of the system. For example, the ordered pair $(1, 2)$ is a solution of system **(a)**, since

$$\begin{aligned} 1 \cdot (1) + 2 \cdot (2) &= 5 \\ 2 \cdot (1) + 3 \cdot (2) &= 8 \end{aligned}$$

The ordered triple $(2, 0, 0)$ is a solution of system **(b)**, since

$$\begin{aligned} 1 \cdot (2) - 1 \cdot (0) + 1 \cdot (0) &= 2 \\ 2 \cdot (2) + 1 \cdot (0) - 1 \cdot (0) &= 4 \end{aligned}$$

Actually, system **(b)** has many solutions. If α is any real number, it is easily seen that the ordered triple $(2, \alpha, \alpha)$ is a solution. However, system **(c)** has no solution. It follows from the third equation that the first coordinate of any solution would have to be 4. Using $x_1 = 4$ in the first two equations, we see that the second coordinate must satisfy

$$\begin{aligned} 4 + x_2 &= 2 \\ 4 - x_2 &= 1 \end{aligned}$$

Since there is no real number that satisfies both of these equations, the system has no solution. If a linear system has no solution, we say that the system is *inconsistent*. If the system has at least one solution, we say that it is *consistent*. Thus system **(c)** is inconsistent, while systems **(a)** and **(b)** are both consistent.

The set of all solutions of a linear system is called the *solution set* of the system. If a system is inconsistent, its solution set is empty. A consistent system will have a nonempty solution set. To solve a consistent system, we must find its solution set.

2×2 Systems

Let us examine geometrically a system of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

Each equation can be represented graphically as a line in the plane. The ordered pair (x_1, x_2) will be a solution of the system if and only if it lies on both lines. For example, consider the three systems

$$\textbf{(i)}\ \begin{aligned} x_1 + x_2 &= 2 \\ x_1 - x_2 &= 2 \end{aligned} \qquad \textbf{(ii)}\ \begin{aligned} x_1 + x_2 &= 2 \\ x_1 + x_2 &= 1 \end{aligned} \qquad \textbf{(iii)}\ \begin{aligned} x_1 + x_2 &= 2 \\ -x_1 - x_2 &= -2 \end{aligned}$$

The two lines in system (i) intersect at the point $(2, 0)$. Thus, $\{(2, 0)\}$ is the solution set of (i). In system (ii) the two lines are parallel. Therefore, system (ii) is inconsistent and hence its solution set is empty. The two equations in system (iii) both represent the same line. Any point on this line will be a solution of the system (see Figure 1.1.1).

In general, there are three possibilities: the lines intersect at a point, they are parallel, or both equations represent the same line. The solution set then contains either one, zero, or infinitely many points.

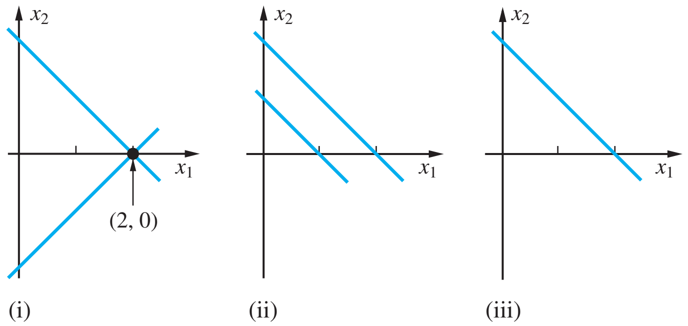


Figure 1.1.1.

The situation is the same for $m \times n$ systems. An $m \times n$ system may or may not be consistent. If it is consistent, it must have either exactly one solution or infinitely many solutions. These are the only possibilities. We will see why this is so in Section 1.2 when we study the row echelon form. Of more immediate concern is the problem of finding all solutions of a given system. To tackle this problem, we introduce the notion of *equivalent systems*.

Equivalent Systems

Consider the two systems

$$\textbf{(a)}\quad \begin{aligned} 3x_1 + 2x_2 - x_3 &= -2 \\ x_2 \phantom{{}- x_3} &= 3 \\ 2x_3 &= 4 \end{aligned} \qquad\qquad \textbf{(b)}\quad \begin{aligned} 3x_1 + 2x_2 - x_3 &= -2 \\ -3x_1 - x_2 + x_3 &= 5 \\ 3x_1 + 2x_2 + x_3 &= 2 \end{aligned}$$

System **(a)** is easy to solve because it is clear from the last two equations that $x_2 = 3$ and $x_3 = 2$. Using these values in the first equation, we get

$$\begin{aligned} 3x_1 + 2 \cdot 3 - 2 &= -2 \\ x_1 &= -2 \end{aligned}$$

Thus, the solution of the system is $(-2, 3, 2)$. System **(b)** seems to be more difficult to solve. Actually, system **(b)** has the same solution as system **(a)**. To see this, add the first two equations of the system:

$$\begin{array}{r} 3x_1 + 2x_2 - x_3 = -2 \\ -3x_1 - x_2 + x_3 = 5 \\ \hline x_2 \phantom{{}+ x_3} = 3 \end{array}$$

If (x_1, x_2, x_3) is any solution of **(b)**, it must satisfy all the equations of the system. Thus, it must satisfy any new equation formed by adding two of its equations. Therefore, x_2 must equal 3. Similarly, (x_1, x_2, x_3) must satisfy the new equation formed by subtracting the first equation from the third:

$$\begin{array}{r} 3x_1 + 2x_2 + x_3 = 2 \\ 3x_1 + 2x_2 - x_3 = -2 \\ \hline 2x_3 = 4 \end{array}$$

Therefore, any solution of system **(b)** must also be a solution of system **(a)**. By a similar argument, it can be shown that any solution of **(a)** is also a solution of **(b)**. This can be done by subtracting the first equation from the second:

$$\begin{array}{rcl} x_2 \qquad\quad &=& 3 \\ 3x_1 + 2x_2 - x_3 &=& -2 \\ \hline -3x_1 - x_2 + x_3 &=& 5 \end{array}$$

Then add the first and third equations:

$$\begin{array}{rcl} 3x_1 + 2x_2 - x_3 &=& -2 \\ 2x_3 &=& 4 \\ \hline 3x_1 + 2x_2 + x_3 &=& 2 \end{array}$$

Thus, (x_1, x_2, x_3) is a solution of system **(b)** if and only if it is a solution of system **(a)**. Therefore, both systems have the same solution set, $\{(-2, 3, 2)\}$.

Definition

Two systems of equations involving the same variables are said to be **equivalent** if they have the same solution set.

Clearly, if we interchange the order in which two equations of a system are written, this will have no effect on the solution set. The reordered system will be equivalent to the original system. For example, the systems

$$\begin{array}{rcl} x_1 + 2x_2 = 4 \\ 3x_1 - x_2 = 2 \\ 4x_1 + x_2 = 6 \end{array} \qquad \text{and} \qquad \begin{array}{rcl} 4x_1 + x_2 = 6 \\ 3x_1 - x_2 = 2 \\ x_1 + 2x_2 = 4 \end{array}$$

both involve the same three equations and, consequently, they must have the same solution set.

If one equation of a system is multiplied through by a nonzero real number, this will have no effect on the solution set, and the new system will be equivalent to the original system. For example, the systems

$$\begin{array}{r} x_1 + x_2 + x_3 = 3 \\ -2x_1 - x_2 + 4x_3 = 1 \end{array} \qquad \text{and} \qquad \begin{array}{r} 2x_1 + 2x_2 + 2x_3 = 6 \\ -2x_1 - x_2 + 4x_3 = 1 \end{array}$$

are equivalent.

If a multiple of one equation is added to another equation, the new system will be equivalent to the original system. This follows since the n-tuple $(x_1, \ldots, x_n)$ will satisfy the two equations

$$\begin{aligned} a_{i1}x_1 + \cdots + a_{in}x_n &= b_i \\ a_{j1}x_1 + \cdots + a_{jn}x_n &= b_j \end{aligned}$$

if and only if it satisfies the equations

$$\begin{aligned} a_{i1}x_1 + \cdots + a_{in}x_n &= b_i \\ (a_{j1} + \alpha a_{i1})x_1 + \cdots + (a_{jn} + \alpha a_{in})x_n &= b_j + \alpha b_i \end{aligned}$$

To summarize, there are three operations that can be used on a system to obtain an equivalent system:

I. The order in which any two equations are written may be interchanged.
II. Both sides of an equation may be multiplied by the same nonzero real number.
III. A multiple of one equation may be added to (or subtracted from) another.

Given a system of equations, we may use these operations to obtain an equivalent system that is easier to solve.

$n \times n$ Systems

Let us restrict ourselves to $n \times n$ systems for the remainder of this section. We will show that if an $n \times n$ system has exactly one solution, then operations **I** and **III** can be used to obtain an equivalent "strictly triangular system."

Definition

> A system is said to be in **strict triangular form** if, in the kth equation, the coefficients of the first $k-1$ variables are all zero and the coefficient of x_k is nonzero $(k = 1, \ldots, n)$.

EXAMPLE 1 The system

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 1 \\ x_2 - x_3 &= 2 \\ 2x_3 &= 4 \end{aligned}$$

is in strict triangular form, since in the second equation the coefficients are 0, 1, -1, respectively, and in the third equation the coefficients are 0, 0, 2, respectively. Because of the strict triangular form, the system is easy to solve. It follows from the third equation that $x_3 = 2$. Using this value in the second equation, we obtain

$$x_2 - 2 = 2 \qquad \text{or} \qquad x_2 = 4$$

Using $x_2 = 4$, $x_3 = 2$ in the first equation, we end up with

$$\begin{aligned} 3x_1 + 2 \cdot 4 + 2 &= 1 \\ x_1 &= -3 \end{aligned}$$

Thus, the solution of the system is $(-3, 4, 2)$. ■

Any $n \times n$ strictly triangular system can be solved in the same manner as the last example. First, the nth equation is solved for the value of x_n. This value is used in the $(n-1)$st equation to solve for x_{n-1}. The values x_n and x_{n-1} are used in the $(n-2)$nd equation to solve for x_{n-2}, and so on. We will refer to this method of solving a strictly triangular system as *back substitution*.

EXAMPLE 2 Solve the system

$$\begin{aligned} 2x_1 - x_2 + 3x_3 - 2x_4 &= 1 \\ x_2 - 2x_3 + 3x_4 &= 2 \\ 4x_3 + 3x_4 &= 3 \\ 4x_4 &= 4 \end{aligned}$$

Solution

Using back substitution, we obtain

$$\begin{aligned} 4x_4 &= 4 & x_4 &= 1 \\ 4x_3 + 3 \cdot 1 &= 3 & x_3 &= 0 \\ x_2 - 2 \cdot 0 + 3 \cdot 1 &= 2 & x_2 &= -1 \\ 2x_1 - (-1) + 3 \cdot 0 - 2 \cdot 1 &= 1 & x_1 &= 1 \end{aligned}$$

Thus the solution is $(1, -1, 0, 1)$. ■

In general, given a system of n linear equations in n unknowns, we will use operations **I** and **III** to try to obtain an equivalent system that is strictly triangular. (We will see in the next section of the book that it is not possible to reduce the system to strictly triangular form in the cases where the system does not have a unique solution.)

EXAMPLE 3 Solve the system

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ 3x_1 - x_2 - 3x_3 &= -1 \\ 2x_1 + 3x_2 + x_3 &= 4 \end{aligned}$$

Solution

Subtracting 3 times the first row from the second row yields

$$-7x_2 - 6x_3 = -10$$

Subtracting 2 times the first row from the third row yields

$$-x_2 - x_3 = -2$$

If the second and third equations of our system, respectively, are replaced by these new equations, we obtain the equivalent system

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ -7x_2 - 6x_3 &= -10 \\ -x_2 - x_3 &= -2 \end{aligned}$$

If the third equation of this system is replaced by the sum of the third equation and $-\frac{1}{7}$ times the second equation, we end up with the following strictly triangular system:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ -7x_2 - 6x_3 &= -10 \\ -\tfrac{1}{7}x_3 &= -\tfrac{4}{7} \end{aligned}$$

Using back substitution, we get

$$x_3 = 4, \qquad x_2 = -2, \qquad x_1 = 3$$

Let us look back at the system of equations in the last example. We can associate with that system a 3×3 array of numbers whose entries are the coefficients of the x_i's:

$$\begin{Bmatrix} 1 & 2 & 1 \\ 3 & -1 & -3 \\ 2 & 3 & 1 \end{Bmatrix}$$

We will refer to this array as the *coefficient matrix* of the system. The term *matrix* means simply a rectangular array of numbers. A matrix having m rows and n columns is said to be $m \times n$. A matrix is said to be *square* if it has the same number of rows and columns, that is, if $m = n$.

If we attach to the coefficient matrix an additional column whose entries are the numbers on the right-hand side of the system, we obtain the new matrix

$$\left\{\begin{array}{rrr|r} 1 & 2 & 1 & 3 \\ 3 & -1 & -3 & -1 \\ 2 & 3 & 1 & 4 \end{array}\right\}$$

We will refer to this new matrix as the *augmented matrix*. In general, when an $m \times r$ matrix B is attached to an $m \times n$ matrix A in this way, the augmented matrix is denoted by $(A|B)$. Thus, if

$$A = \begin{Bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{Bmatrix}, \qquad B = \begin{Bmatrix} b_{11} & b_{12} & \cdots & b_{1r} \\ b_{21} & b_{22} & \cdots & b_{2r} \\ \vdots & & & \\ b_{m1} & b_{m2} & \cdots & b_{mr} \end{Bmatrix}$$

then

$$(A|B) = \left\{\begin{array}{ccc|ccc} a_{11} & \cdots & a_{1n} & b_{11} & \cdots & b_{1r} \\ \vdots & & & \vdots & & \\ a_{m1} & \cdots & a_{mn} & b_{m1} & \cdots & b_{mr} \end{array}\right\}$$

With each system of equations we may associate an augmented matrix of the form

$$\left\{\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right\}$$

The system can be solved by performing operations on the augmented matrix. The x_i's are placeholders that can be omitted until the end of the computation. Corresponding to the three operations used to obtain equivalent systems, the following row operations may be applied to the augmented matrix:

Elementary Row Operations

I. Interchange two rows.
II. Multiply a row by a nonzero real number.
III. Replace a row by its sum with a multiple of another row.

Returning to the example, we find that the first row is used to eliminate the elements in the first column of the remaining rows. We refer to the first row as the *pivotal row*. For emphasis, the entries in the pivotal row are all in bold type and the entire row is color shaded. The first nonzero entry in the pivotal row is called the *pivot*.

$$\begin{array}{r}(\text{pivot } a_{11} = 1)\\ \left.\begin{array}{r}\text{entries to be eliminated}\\ a_{21} = 3 \text{ and } a_{31} = 2\end{array}\right\}\rightarrow\end{array} \left[\begin{array}{rrr|r} \mathbf{1} & \mathbf{2} & \mathbf{1} & \mathbf{3} \\ 3 & -1 & -3 & -1 \\ 2 & 3 & 1 & 4 \end{array}\right] \begin{array}{l}\leftarrow \text{pivotal row}\\ \\ \\ \end{array}$$

By using row operation III, 3 times the first row is subtracted from the second row and 2 times the first row is subtracted from the third. When this is done, we end up with the matrix

$$\left[\begin{array}{rrr|r} 1 & 2 & 1 & 3 \\ \mathbf{0} & \mathbf{-7} & \mathbf{-6} & \mathbf{-10} \\ 0 & -1 & -1 & -2 \end{array}\right] \begin{array}{l} \\ \leftarrow \text{pivotal row} \\ \\ \end{array}$$

At this step we choose the second row as our new pivotal row and apply row operation III to eliminate the last element in the second column. This time the pivot is -7 and the quotient $\frac{-1}{-7} = \frac{1}{7}$ is the multiple of the pivotal row that is subtracted from the third row. We end up with the matrix

$$\left[\begin{array}{rrr|r} 1 & 2 & 1 & 3 \\ 0 & -7 & -6 & -10 \\ 0 & 0 & -\frac{1}{7} & -\frac{4}{7} \end{array}\right]$$

This is the augmented matrix for the strictly triangular system, which is equivalent to the original system. The solution of the system is easily obtained by back substitution.

EXAMPLE 4 Solve the system

$$\begin{aligned} - x_2 - x_3 + x_4 &= 0 \\ x_1 + x_2 + x_3 + x_4 &= 6 \\ 2x_1 + 4x_2 + x_3 - 2x_4 &= -1 \\ 3x_1 + x_2 - 2x_3 + 2x_4 &= 3 \end{aligned}$$

Solution

The augmented matrix for this system is

$$\left[\begin{array}{rrrr|r} 0 & -1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 6 \\ 2 & 4 & 1 & -2 & -1 \\ 3 & 1 & -2 & 2 & 3 \end{array}\right]$$

Since it is not possible to eliminate any entries by using 0 as a pivot element, we will use row operation I to interchange the first two rows of the augmented matrix. The new first row will be the pivotal row and the pivot element will be 1:

$$\begin{array}{r}(\text{pivot } a_{11} = 1)\\ \\ \\ \\ \end{array} \left[\begin{array}{rrrr|r} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{6} \\ 0 & -1 & -1 & 1 & 0 \\ 2 & 4 & 1 & -2 & -1 \\ 3 & 1 & -2 & 2 & 3 \end{array}\right] \begin{array}{l}\leftarrow \text{pivotal row}\\ \\ \\ \\ \end{array}$$

Row operation III is then used twice to eliminate the two nonzero entries in the first column:

$$\left(\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 6 \\ \mathbf{0} & \mathbf{-1} & \mathbf{-1} & \mathbf{1} & \mathbf{0} \\ 0 & 2 & -1 & -4 & -13 \\ 0 & -2 & -5 & -1 & -15 \end{array}\right)$$

Next, the second row is used as the pivotal row to eliminate the entries in the second column below the pivot element -1:

$$\left(\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & -1 & 1 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{-3} & \mathbf{-2} & \mathbf{-13} \\ 0 & 0 & -3 & -3 & -15 \end{array}\right)$$

Finally, the third row is used as the pivotal row to eliminate the last element in the third column:

$$\left(\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & -3 & -2 & -13 \\ 0 & 0 & 0 & -1 & -2 \end{array}\right)$$

This augmented matrix represents a strictly triangular system. Solving by back substitution, we obtain the solution $(2, -1, 3, 2)$. ■

In general, if an $n \times n$ linear system can be reduced to strictly triangular form, then it will have a unique solution that can be obtained by performing back substitution on the triangular system. We can think of the reduction process as an algorithm involving $n-1$ steps. At the first step, a pivot element is chosen from among the nonzero entries in the first column of the matrix. The row containing the pivot element is called the *pivotal row*. We interchange rows (if necessary) so that the pivotal row is the new first row. Multiples of the pivotal row are then subtracted from each of the remaining $n-1$ rows so as to obtain 0's in the first entries of rows 2 through n. At the second step, a pivot element is chosen from the nonzero entries in column 2, rows 2 through n, of the matrix. The row containing the pivot is then interchanged with the second row of the matrix and is used as the new pivotal row. Multiples of the pivotal row are then subtracted from the remaining $n-2$ rows so as to eliminate all entries below the pivot in the second column. The same procedure is repeated for columns 3 through $n-1$. Note that at the second step row 1 and column 1 remain unchanged, at the third step the first two rows and first two columns remain unchanged, and so on. At each step, the overall dimensions of the system are effectively reduced by 1 (see Figure 1.1.2).

If the elimination process can be carried out as described, we will arrive at an equivalent strictly triangular system after $n - 1$ steps. However, the procedure will break down if, at any step, all possible choices for a pivot element are equal to 0. When this happens, the alternative is to reduce the system to certain special echelon, or staircase-shaped, forms. These echelon forms will be studied in the next section. They will also be used for $m \times n$ systems, where $m \neq n$.

Step 1
$$\left(\begin{array}{cccc|c} x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} x & x & x & x & x \\ 0 & x & x & x & x \\ 0 & x & x & x & x \\ 0 & x & x & x & x \end{array}\right)$$

Step 2
$$\left(\begin{array}{cccc|c} x & x & x & x & x \\ 0 & x & x & x & x \\ 0 & x & x & x & x \\ 0 & x & x & x & x \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} x & x & x & x & x \\ 0 & x & x & x & x \\ 0 & 0 & x & x & x \\ 0 & 0 & x & x & x \end{array}\right)$$

Step 3
$$\left(\begin{array}{cccc|c} x & x & x & x & x \\ 0 & x & x & x & x \\ 0 & 0 & x & x & x \\ 0 & 0 & x & x & x \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} x & x & x & x & x \\ 0 & x & x & x & x \\ 0 & 0 & x & x & x \\ 0 & 0 & 0 & x & x \end{array}\right)$$

Figure 1.1.2.

SECTION 1.1 EXERCISES

1. Use back substitution to solve each of the following systems of equations:

(a) $$\begin{aligned} x_1 - 3x_2 &= 2 \\ 2x_2 &= 6 \end{aligned}$$

(b) $$\begin{aligned} x_1 + x_2 + x_3 &= 8 \\ 2x_2 + x_3 &= 5 \\ 3x_3 &= 9 \end{aligned}$$

(c) $$\begin{aligned} x_1 + 2x_2 + 2x_3 + x_4 &= 5 \\ 3x_2 + x_3 - 2x_4 &= 1 \\ -x_3 + 2x_4 &= -1 \\ 4x_4 &= 4 \end{aligned}$$

(d) $$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 5 \\ 2x_2 + x_3 - 2x_4 + x_5 &= 1 \\ 4x_3 + x_4 - 2x_5 &= 1 \\ x_4 - 3x_5 &= 0 \\ 2x_5 &= 2 \end{aligned}$$

2. Write out the coefficient matrix for each of the systems in Exercise 1.

3. In each of the following systems, interpret each equation as a line in the plane. For each system, graph the lines and determine geometrically the number of solutions.

(a) $$\begin{aligned} x_1 + x_2 &= 4 \\ x_1 - x_2 &= 2 \end{aligned}$$

(b) $$\begin{aligned} x_1 + 2x_2 &= 4 \\ -2x_1 - 4x_2 &= 4 \end{aligned}$$

(c) $$\begin{aligned} 2x_1 - x_2 &= 3 \\ -4x_1 + 2x_2 &= -6 \end{aligned}$$

(d) $$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 - x_2 &= 1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

4. Write an augmented matrix for each of the systems in Exercise 3.

5. Write out the system of equations that corresponds to each of the following augmented matrices:

(a) $$\left(\begin{array}{cc|c} 3 & 2 & 8 \\ 1 & 5 & 7 \end{array}\right)$$

(b) $$\left(\begin{array}{ccc|c} 5 & -2 & 1 & 3 \\ 2 & 3 & -4 & 0 \end{array}\right)$$

(c) $$\left(\begin{array}{ccc|c} 2 & 1 & 4 & -1 \\ 4 & -2 & 3 & 4 \\ 5 & 2 & 6 & -1 \end{array}\right)$$

(d) $$\left(\begin{array}{cccc|c} 4 & -3 & 1 & 2 & 4 \\ 3 & 1 & -5 & 6 & 5 \\ 1 & 1 & 2 & 4 & 8 \\ 5 & 1 & 3 & -2 & 7 \end{array}\right)$$

6. Solve each of the following systems.

(a) $$\begin{aligned} x_1 - 2x_2 &= 5 \\ 3x_1 + x_2 &= 1 \end{aligned}$$

(b) $$\begin{aligned} 2x_1 + x_2 &= 8 \\ 4x_1 - 3x_2 &= 6 \end{aligned}$$

(c) $$\begin{aligned} 4x_1 + 3x_2 &= 4 \\ \tfrac{2}{3}x_1 + 4x_2 &= 3 \end{aligned}$$

(d) $$\begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ 2x_1 - x_2 + x_3 &= 3 \\ -x_1 + 2x_2 + 3x_3 &= 7 \end{aligned}$$

(e) $$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 1 \\ 4x_1 + 3x_2 + 5x_3 &= 1 \\ 6x_1 + 5x_2 + 5x_3 &= -3 \end{aligned}$$

(f) $$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 0 \\ -2x_1 + x_2 - x_3 &= 2 \\ 2x_1 - x_2 + 2x_3 &= -1 \end{aligned}$$

(g) $$\begin{aligned} \tfrac{1}{3}x_1 + \tfrac{2}{3}x_2 + 2x_3 &= -1 \\ x_1 + 2x_2 + \tfrac{3}{2}x_3 &= \tfrac{3}{2} \\ \tfrac{1}{2}x_1 + 2x_2 + \tfrac{12}{5}x_3 &= \tfrac{1}{10} \end{aligned}$$

(h) $$\begin{aligned} x_2 + x_3 + x_4 &= 0 \\ 3x_1 + 3x_3 - 4x_4 &= 7 \\ x_1 + x_2 + x_3 + 2x_4 &= 6 \\ 2x_1 + 3x_2 + x_3 + 3x_4 &= 6 \end{aligned}$$

7. The two systems

$$\begin{aligned} 2x_1 + x_2 &= 3 \\ 4x_1 + 3x_2 &= 5 \end{aligned} \quad \text{and} \quad \begin{aligned} 2x_1 + x_2 &= -1 \\ 4x_1 + 3x_2 &= 1 \end{aligned}$$

have the same coefficient matrix but different right-hand sides. Solve both systems simultaneously by eliminating the first entry in the second row of the augmented matrix

$$\left[\begin{array}{cc|cc} 2 & 1 & 3 & -1 \\ 4 & 3 & 5 & 1 \end{array}\right]$$

and then performing back substitutions for each of the columns corresponding to the right-hand sides.

8. Solve the two systems

$$\begin{aligned} x_1 + 2x_2 - 2x_3 &= 1 \\ 2x_1 + 5x_2 + x_3 &= 9 \\ x_1 + 3x_2 + 4x_3 &= 9 \end{aligned} \qquad \begin{aligned} x_1 + 2x_2 - 2x_3 &= 9 \\ 2x_1 + 5x_2 + x_3 &= 9 \\ x_1 + 3x_2 + 4x_3 &= -2 \end{aligned}$$

by doing elimination on a 3×5 augmented matrix and then performing two back substitutions.

9. Given a system of the form

$$\begin{aligned} -m_1x_1 + x_2 &= b_1 \\ -m_2x_1 + x_2 &= b_2 \end{aligned}$$

where m_1, m_2, b_1, and b_2 are constants:

(a) Show that the system will have a unique solution if $m_1 \neq m_2$.

(b) Show that if $m_1 = m_2$, then the system will be consistent only if $b_1 = b_2$.

(c) Give a geometric interpretation of parts (a) and (b).

10. Consider a system of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= 0 \\ a_{21}x_1 + a_{22}x_2 &= 0 \end{aligned}$$

where a_{11}, a_{12}, a_{21}, and a_{22} are constants. Explain why a system of this form must be consistent.

11. Give a geometrical interpretation of a linear equation in three unknowns. Give a geometrical description of the possible solution sets for a 3×3 linear system.

1.2 Row Echelon Form

In Section 1.1 we learned a method for reducing an $n \times n$ linear system to strict triangular form. However, this method will fail if, at any stage of the reduction process, all the possible choices for a pivot element in a given column are 0.

EXAMPLE 1 Consider the system represented by the augmented matrix

$$\left[\begin{array}{rrrrr|r} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ -1 & -1 & 0 & 0 & 1 & -1 \\ -2 & -2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 1 & 1 & 2 & 2 & 4 & 1 \end{array}\right] \begin{array}{l} \leftarrow \text{pivotal row} \\ \\ \\ \\ \\ \end{array}$$

If row operation III is used to eliminate the nonzero entries in the last four rows of the first column, the resulting matrix will be

$$\left[\begin{array}{rrrrr|r} 1 & 1 & 1 & 1 & 1 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{2} & \mathbf{0} \\ 0 & 0 & 2 & 2 & 5 & 3 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 1 & 1 & 3 & 0 \end{array}\right] \begin{array}{l} \\ \leftarrow \text{pivotal row} \\ \\ \\ \\ \end{array}$$

At this stage, the reduction to strict triangular form breaks down. All four possible choices for the pivot element in the second column are 0. How do we proceed from here? Since our goal is to simplify the system as much as possible, it seems natural to move over to the third column and eliminate the last three entries:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{3} \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

In the fourth column, all the choices for a pivot element are 0; so again we move on to the next column. If we use the third row as the pivotal row, the last two entries in the fifth column are eliminated and we end up with the matrix

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{array}\right]$$

The coefficient matrix that we end up with is not in strict triangular form; it is in staircase, or echelon, form. The horizontal and vertical line segments in the array for the coefficient matrix indicate the structure of the staircase form. Note that the vertical drop is 1 for each step, but the horizontal span for a step can be more than 1.

The equations represented by the last two rows are

$$\begin{aligned} 0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 &= -4 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 &= -3 \end{aligned}$$

Since there are no 5-tuples that could satisfy these equations, the system is inconsistent. ■

Suppose now that we change the right-hand side of the system in the last example so as to obtain a consistent system. For example, if we start with

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & -1 \\ -2 & -2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 3 & 3 \\ 1 & 1 & 2 & 2 & 4 & 4 \end{array}\right]$$

then the reduction process will yield the echelon-form augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The last two equations of the reduced system will be satisfied for any 5-tuple. Thus the solution set will be the set of all 5-tuples satisfying the first three equations.

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 1 \\ x_3 + x_4 + 2x_5 &= 0 \\ x_5 &= 3 \end{aligned} \tag{1}$$

The variables corresponding to the first nonzero elements in each row of the reduced matrix will be referred to as *lead variables*. Thus x_1, x_3, and x_5 are the lead variables. The remaining variables corresponding to the columns skipped in the reduction process will be referred to as *free variables*. Hence, x_2 and x_4 are the free variables. If we transfer the free variables over to the right-hand side in (1), we obtain the system

$$\begin{aligned} x_1 + x_3 + x_5 &= 1 - x_2 - x_4 \\ x_3 + 2x_5 &= -x_4 \\ x_5 &= 3 \end{aligned} \tag{2}$$

System (2) is strictly triangular in the unknowns x_1, x_3, and x_5. Thus, for each pair of values assigned to x_2 and x_4, there will be a unique solution. For example, if $x_2 = x_4 = 0$, then $x_5 = 3$, $x_3 = -6$, and $x_1 = 4$, and hence $(4, 0, -6, 0, 3)$ is a solution of the system.

Definition

A matrix is said to be in **row echelon form** if

(i) The first nonzero entry in each nonzero row is 1.

(ii) If row k does not consist entirely of zeros, the number of leading zero entries in row $k + 1$ is greater than the number of leading zero entries in row k.

(iii) If there are rows whose entries are all zero, they are below the rows having nonzero entries.

EXAMPLE 2 The following matrices are in row echelon form:

$$\begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

■

EXAMPLE 3 The following matrices are not in row echelon form:

$$\begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 5 \\ 0 & 0 & 4 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The first matrix does not satisfy condition (i). The second matrix fails to satisfy condition (iii), and the third matrix fails to satisfy condition (ii). ■

Definition

The process of using row operations I, II, and III to transform a linear system into one whose augmented matrix is in row echelon form is called **Gaussian elimination**.

Note that row operation II is necessary in order to scale the rows so that the leading coefficients are all 1. If the row echelon form of the augmented matrix contains a row of the form

$$\left(\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & 1 \end{array}\right)$$

the system is inconsistent. Otherwise, the system will be consistent. If the system is consistent and the nonzero rows of the row echelon form of the matrix form a strictly triangular system, the system will have a unique solution.

Overdetermined Systems

A linear system is said to be *overdetermined* if there are more equations than unknowns. Overdetermined systems are *usually* (but not always) inconsistent.

EXAMPLE 4 Solve each of the following overdetermined systems:

(a)
$$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 - x_2 &= 3 \\ -x_1 + 2x_2 &= -2 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 2x_1 - x_2 + x_3 &= 2 \\ 4x_1 + 3x_2 + 3x_3 &= 4 \\ 2x_1 - x_2 + 3x_3 &= 5 \end{aligned}$$

(c)
$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 2x_1 - x_2 + x_3 &= 2 \\ 4x_1 + 3x_2 + 3x_3 &= 4 \\ 3x_1 + x_2 + 2x_3 &= 3 \end{aligned}$$

Solution

By now the reader should be familiar enough with the elimination process that we can omit the intermediate steps in reducing each of these systems. Thus, we may write

$$\text{System (a):} \quad \left(\begin{array}{rr|r} 1 & 1 & 1 \\ 1 & -1 & 3 \\ -1 & 2 & -2 \end{array}\right) \rightarrow \left(\begin{array}{rr|r} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{array}\right)$$

It follows from the last row of the reduced matrix that the system is inconsistent. The three equations in system (a) represent lines in the plane. The first two lines intersect at the point $(2, -1)$. However, the third line does not pass through this point. Thus, there are no points that lie on all three lines (see Figure 1.2.1).

$$\text{System (b):} \quad \left(\begin{array}{rrr|r} 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 2 \\ 4 & 3 & 3 & 4 \\ 2 & -1 & 3 & 5 \end{array}\right) \rightarrow \left(\begin{array}{rrr|r} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{1}{5} & 0 \\ 0 & 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Using back substitution, we see that system (b) has exactly one solution $(0.1, -0.3, 1.5)$. The solution is unique because the nonzero rows of the reduced matrix